Differential Regularization of Topologically Massive Yang-Mills Theory and Chern-Simons Theory

W.F. Chen[d] [†], H.C. Lee[e] and Z.Y. Zhu[a,b,c]

[a] *CCAST (World Laboratory), P.O. Box 8730, Beijing, 100080, China*

[b] *Institute of Theoretical Physics, Academia Sinica,*

P. O. Box 2735, Beijing, 100080, China

[c] *Department of Physics, Chung-Hsing University, Taichung, Taiwan, China*

[d] *Research Institute for High Energy Physics, University of Helsinki,*

P.O. Box 9 (Siltavuorenpenger 20 C), FIN-000014, Helsinki, Finland

[e] *Department of Physics, National Central University, Chungli, Taiwan 320, China*

Abstract

We apply differential renormalization method to the study of three-dimensional topologically massive Yang-Mills and Chern-Simons theories. The method is especially suitable for such theories as it avoids the need for dimensional continuation of three-dimensional antisymmetric tensor and the Feynman rules for three-dimensional theories in coordi-

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[†]ICSC-World Laboratory, Switzerland

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nate space are relatively simple. The calculus involved is still lengthy but not as difficult as other existing methods of calculation. We compute one-loop propagators and vertices and derive the one-loop local effective action for topologically massive Yang-Mills theory. We then consider Chern-Simons field theory as the large mass limit of topologically massive Yang-Mills theory and show that this leads to the famous shift in the parameter k. Some useful formulas for the calculus of differential renormalization of three-dimensional field theories are given in an Appendix.

I. INTRODUCTION

The differential renormalization (DR) method was proposed by Freedman, Johnson and Latorre [1] to deal with the ultraviolet divergences of quantum field theories few years ago. Its original idea came from the observation that primitively divergent amplitudes are well defined in coordinate space for non-coindent points, but too singular at short distance to allow a Fourier transform into momentum space. They proposed to renormalize such an amplitude by first writing its singular parts as derivatives of some less singular functions that have well defined Fourier transformations, then performing Fourier transformations of such functions and discarding the surface terms. This idea is clearly illustrated when one applies it to the one-loop

4-point bubble graph of massless ϕ^4 theory in 4-dimensional space-time. As we know, the amplitude of this graph involves the function $1/x^4$ that is singular at $x = 0$, corresponding to an ultraviolet divergence. To realize its differential renormalization we follow ref. [1] and use the identity,

$$\frac{1}{x^4} = -\frac{1}{4} \Box \frac{\ln(x^2 M^2)}{x^2}, \quad \text{for } x \neq 0. \tag{1}$$

The function $\ln(x^2 M^2)/x^2$ has a well defined Fourier transform $4\pi^2 \ln(p^2/\overline{M}^2)/p^2$, where $\overline{M} = M/\gamma$ and γ is Euler's constant. After discarding the surface term we are left with $-\pi^2 \ln(p^2/\overline{M}^2)$ as the regulated Fourier transform of $1/x^4$.

The DR method has been applied to many cases including massless ϕ^4 theory up to three-loop order [1], one-loop massive ϕ^4 theory [2], supersymmetric Wess-Zumino model up to three loops [3], Yang-Mills theory in background field method up to one-loop [1], QED up to two loops [4] and low dimensional Abelian gauge theories to one-loop [5]. Its relation with the conventional dimensional regularization in some theories [6, 22] and compatibility with unitary have been investigated [7] and it has been shown to be simpler and more powerful than other regularizations in many cases.

In this paper we shall use the DR method to study the perturbative three dimensional topologically massive Yang-Mills theory (TMYM) and Chern-Simons theory (CS) which, as will be shown, it is especially suited for.

The action of TMYM [8], which is obtained by adding to the standard non-Abelian gauge action, the Chern-Simons term, can be written in Euclidean space as,

$$S_m = -i\frac{k}{4\pi} \int_x \epsilon^{\mu\nu\rho} \left(\frac{1}{2} A_\mu^a \partial_\nu A_\rho^a + \frac{1}{3!} f^{abc} A_\mu^a A_\nu^b A_\rho^c \right) + \frac{|k|}{16m\pi^2} \int_x F_{\mu\nu}^a F^{\mu\nu a}, \tag{2}$$

where the integration $\int_x \equiv \int d^3x$ is over the whole R^3. The first term, i.e., the

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Chern-Simons term, exists only in three dimensions. It is easy to see that under a gauge transformation U the action transforms as

$$S_m \rightarrow S_m - 2\pi i k S_{WZ}\,,$$

$$S_{WZ} = \frac{1}{24\pi^2} \int_x \epsilon^{\mu\nu\rho} \operatorname{Tr}\left(U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\right)\,.$$

As it is well known, the Wess-Zumino term S_{WZ} takes integer value, so the theory is expected to be gauge invariant at the quantum level when k takes integer value. At the same time, an interesting property of S_m is that the gauge excitations are massive, with mass m. This property exits only in three dimensions and is not shared by other dimensions.

The perturbative property of TMYM was studied in [8,9], where it was pointed out that the computations involved are not trivial and require diligence. In our view, dimensionality plays an important role in defining three-dimensional TMYM because much of the topological properties of the theory are derived from the properties of three dimensional antisymmetric tensor $\epsilon^{\mu\nu\rho}$. A calculation without using dimensional continuation is therefore called for. We are furthermore motivated to study the theory with DR by the fact that in three dimensions, propagators of TMYM in coordinate space have analytic forms that are particularly suited for the application of this method.

During the past several years a number of studies of perturbative Chern-Simons theory have been carried out using a variety of regularization schemes including: higher covariant derivative (HCD) combined with generalized Pauli-Villars regularization [10]; HCD combined with dimensional regularization [11,12]; operator regularization [13]; η function regularization [14]; geometric regularization [15] and Feynman propagator regularization [16]. Especially recently an understanding on the perturbative behaviour of CS from supersymmetric Yang-Mills-Chern-Simons theory

has appeared [17] and a more strict mathematical treatment from the geometric viewpoint has been discussed in ref. [18]. From these studies there emerges the so-called k-shift problem in three-dimensional CS, which is concerned with whether quantum correction change the value of the parameter k. It appears that whether the value of k shifts or not depends on the regularization scheme [19, 20], – some of the calculations in these studies showed the k-shift while others did not. In ref. [19], an analysis shows a family of shift can be generated, which depends on the parity property of the regulator.

As we know, Chern-Simons action is just the first term in Eq.(2). Obviously we can consider TMYM as a partially (high covariant derivative) regulated version of CS or, alternatively, CS as the large mass limit ($m\rightarrow\infty$) of TMYM [11]. So a calculation of the perturbative property of three-dimensional TMYM yields a study of the k -shift problem of three-dimensional CS as a by-product. Our result confirms the existence of k-shift and coincides with the case of scalar regulators of ref. [19].

This paper is organized as follows. In section II we present the Feynman rules of TMYM in coordinate space. Section III is devoted to explicit calculations of one-loop amplitudes needed for the computation of one-loop local effective action, where we have obtained ghost self-energy, vacuum polarization tensor and gauge boson-ghost-ghost vertex. In Section IV Slavnov-Taylor identity is explicitly derived and used in combination of results from Section III to determine the one-loop local effective action. In section V, as an example demonstrating the usefulness of formulas given in Appendix, we give a result for the self-energy of gauge field in three-dimensional QED. In section VI, we discuss and summarize the results. Some formulas utilized in the calculation are given in the Appendix. These formulas should also be useful for DR calculations of other low-dimensional field theories.

II. FEYNMAN RULES IN COORDINATE SPACE

By defining $g^2 = 4\pi/|k|$ and rescaling $A \to A/g$, we rewrite TMYM action (1) as

$$S_m = -i\,\mathrm{sgn}(k) \int_x \epsilon^{\mu\nu\rho} \left(\frac{1}{2} A_\mu^a \partial_\nu A_\rho^a + \frac{1}{3!} g f^{abc} A_\mu^a A_\nu^b A_\rho^c \right) + \frac{1}{4m} \int_x F_{\mu\nu}^a F^{\mu\nu a} \,, \qquad (3)$$

whose corresponding BRST invariant action in the Landau gauge is

$$S[A, c, \bar{c}, B, m] = S_m + \int_x \left[\partial_\mu \bar{c}^a D^\mu c^a + B^a \partial_\mu A^{\mu a} \right] . \qquad (4)$$

The BRST transformation of the fields are

$$\delta A_\mu^a = D_\mu c^a \,, \qquad \delta \bar{c}^a = B^a \,,$$
$$\delta c^a = -\frac{1}{2} g f^{abc} c^b c^c \,, \qquad \delta B^a = 0 \,. \qquad (5)$$

Here we choose the Landau gauge because of its good infrared behavior [9]. For a pure Chern-Simons field theory, the Landau vector supersymmetry [20, 23, 24],

$$v_\mu A_\nu^a = i\,\mathrm{sgn}(k)\epsilon_{\nu\mu\rho}\partial^\rho c^a \,, \quad v_\mu c^a = 0 \,,$$
$$v_\mu \bar{c}^a = A_\mu^a \,, \quad v_\mu B^a = -D_\mu c^a \,, \qquad (6)$$

which only exists in the Landau gauge, plays a crucial role in the cancellation of the infrared divergence. Although the inclusion of Yang-Mills term in TMYM breaks this symmetry, it does not ruin the cancellation of the infrared singularity.

The generating functional can be formally written as

$$Z[J, \eta, \bar{\eta}, M] = \int \mathcal{D}A\mathcal{D}B\mathcal{D}c\mathcal{D}\bar{c} \, \exp\left(-S - \int \left[J_\mu^a A_\mu^a + \bar{\eta}^a c^a + \bar{c}^a \eta^a + B^a M^a \right] \right) . \qquad (7)$$

Differential regularization works in coordinate space, so we need the Feynman rules in coordinate space. Defining

$$G_{\mu\nu}^{ab}(x - y) = \,< 0|T[A_\mu^a(x)A_\nu^b(y)]|0 > \,,$$
$$\Lambda_\mu^{ab}(x - y) = \,< 0|T[A_\mu^a(x)B^b(y)]|0 > \,,$$

$$\Lambda^{ab}(x-y) = <0|T[B^a(x)B^b(y)]0>,$$

$$S^{ab}(x-y) = <0|T[c^a(x)\bar{c}^b(y)]|0>,$$

$$<0|T[A^a_\mu(x)A^b_\nu(y)A^c_\rho(y)]|0> = \int_{x'}\int_{y'}\int_{z'} G^{aa'}_{\mu\mu'}(x-x')G^{bb'}_{\nu\nu'}(y-y')G^{cc'}_{\rho\rho'}(z-z')$$
$$\times \Gamma^{a'b'c'}_{(3)\mu'\nu'\rho'}(x',y',z'),$$

$$<0|T[A^a_\mu(x)A^b_\nu(y)A^c_\rho(z)A^d_\sigma(w)]|0> = \int_{x'}\int_{y'}\int_{z'} G^{aa'}_{\mu\mu'}(x-x')G^{bb'}_{\nu\nu'}(y-y') \times$$
$$G^{cc'}_{\rho\rho'}(z-z')G^{dd'}_{\sigma\sigma'}(w-w')\Gamma^{a'b'c'd'}_{(4)\mu'\nu'\rho'\sigma'}(x',y',z',w'),$$

$$<0|T[c^a(x)\bar{c}^c(z)A^b_\mu(y)]|0> = \int_{x'}\int_{y'}\int_{z'} S^{aa'}(x-x')G^{bb'}_{\mu\mu'}(y-y')S^{c'c}(z'-z)$$
$$\times \Lambda^{a'b'c'}_{\mu'}(x',y',z'), \tag{8}$$

we obtain Feynman rules as follows (Fig. 1),

$$G^{(0)ab}_{\mu\nu}(x-y) \equiv \delta^{ab}D_{\mu\nu}(x-y)$$
$$= -\delta^{ab}\left[i\,\mathrm{sgn}(k)\epsilon_{\mu\nu\rho}\partial^\rho_x + \frac{1}{m}\left(\delta_{\mu\nu}\nabla^{2^x} - \partial^x_\mu\partial^x_\nu\right)\right]\frac{(1-e^{-m|x-y|})}{4\pi|x-y|},$$

$$\Lambda^{(0)ab}_\mu(x-y) \equiv \delta^{ab}\Lambda_\mu(x-y) = -\delta^{ab}\partial^x_\mu\frac{1}{4\pi|x-y|},$$

$$\Lambda^{(0)ab}(x-y) = 0,$$

$$S^{(0)ab}(x-y) \equiv \delta^{ab}S(x-y) = \delta^{ab}\frac{1}{4\pi|x-y|},$$

$$\Gamma^{(0)abc}_{(3)\mu\nu\rho}(x,y,z) = gf^{abc}\left[i\,\mathrm{sgn}(k)\epsilon_{\mu\nu\rho} - \frac{1}{m}[(\partial^y_\mu - \partial^z_\mu)\delta_{\nu\rho} + (\partial^z_\nu - \partial^x_\nu)\delta_{\rho\mu}\right.$$
$$\left. +(\partial^x_\rho - \partial^y_\rho)\delta_{\mu\nu}\right]\int_u \delta^{(3)}(x-u)\delta^{(3)}(y-u)\delta^{(3)}(z-u),$$

$$\Gamma^{(0)abcd}_{(4)\mu\nu\rho\sigma}(x,y,z,w) = \frac{g^2}{m}\left[f^{eab}f^{ecd}(\delta_{\nu\rho}\delta_{\sigma\mu} - \delta_{\nu\sigma}\delta_{\mu\rho}) + f^{eac}f^{edb}(\delta_{\rho\sigma}\delta_{\nu\mu} - \delta_{\nu\rho}\delta_{\mu\sigma})\right.$$
$$\left. +f^{ead}f^{ebc}(\delta_{\sigma\nu}\delta_{\rho\mu} - \delta_{\sigma\rho}\delta_{\mu\nu})\right]\int_u \delta^{(3)}(x-u)\delta^{(3)}(y-u)\delta^{(3)}(z-u)\delta^{(3)}(w-u),$$

$$\Lambda^{(0)abc}_\mu(x,y,z) = gf^{abc}\partial^x_\mu\int_u \delta^{(3)}(x-u)\delta(y-u)\delta(z-u), \tag{9}$$

where the superscript " (0) " denotes free propagators or bare vertices. We can see that the propagators given above are much simpler in comparison with their counterparts in 4-dimensional massive theories, which are Bessel functions [21].

III. ONE-LOOP AMPLITUDES

Now we use DR to to carry out the one-loop renormalization of TMYM. Naive power counting suggests that some of the one-loop diagrams should be ultraviolet divergent. But as we will show, in some sense, TMYM is essentially a finite theory [9]. Purely for the purpose of making this finiteness manifest (DR does not require it), we introduce a short distance cutoff by excluding a small ball B_ϵ of radius ϵ about the origin as in [1, 7]. Denote the region $R^3 - B_\epsilon$ by R^3_ϵ.

Let us analyze the one-loop ghost self-energy first. Its Feynman diagram is shown in Fig. 2. The Fourier transformation of its amplitude is

$$- g^2 C_V \delta^{ab} \int_{R^3_\epsilon} \partial^x_\mu (e^{-ip.x}) D_{\mu\nu}(x) \partial^x_\nu S(x) . \tag{10}$$

Here we would like to emphasize that we need to be careful about the positions of the partial differential operators in the Feynman rules of (9). By using (9) and the formulas in Appendix we have

$$S^{(1)ab}(p) = \delta^{ab} \frac{g^2 C_V}{8\pi^2 m} \int_{R^3_\epsilon} e^{-ip.x} ip_\mu x_\mu \left[\frac{1 - e^{-mr}}{r^6} - \frac{m e^{-mr}}{r^5} \right] . \tag{11}$$

Writing singular functions at $r = 0$ in the above integrand as derivatives of less singular functions, we get

$$S^{(1)ab}(p) = -\delta^{ab} \frac{g^2 C_V}{8\pi^2 m} \int_{R^3_\epsilon} e^{-ip.x} ip_\mu \partial_\mu \left[\nabla^2 \frac{1 - e^{-mr}}{8r^2} \right.$$
$$\left. + \frac{m^2 e^{-mr}}{4r^2} - \frac{m^3 e^{-mr}}{6r} - \frac{m^4 \text{Ei}(-mr)}{8} \right] , \tag{12}$$

where $\text{Ei}(x)$ is the exponential integral function:

$$\text{Ei}(-mr) = \int_{-\infty}^{-mr} dt \, \frac{e^t}{t} . \tag{13}$$

Particular attentions should be paid to differential operators in (12) when we perform the Fourier transformation. For example,

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$$\int_{R^3_\epsilon} e^{-ip.x}\nabla^2\frac{1-e^{-mr}}{8r^2} = \int_{R^3_\epsilon}\left[\nabla^2\left(e^{-ip.x}\frac{1-e^{-mr}}{8r^2}\right)\right.$$
$$\left. -2\,\partial_\mu e^{-ip.x}\partial_\mu\frac{1-e^{-mr}}{8r^2} + p^2\,e^{-ip.x}\frac{1-e^{-mr}}{8r^2}\right] \tag{14}$$
$$= \frac{m\pi}{2} + p^2\int_{R^3_\epsilon}e^{-ip.x}\frac{1-e^{-mr}}{8r^2}\,,$$

where the first term on the right-hand-side is a surface term from the cut-off ball B_ϵ.

Finally, using the formulas in Appendix, we obtain the one-loop ghost self-energy

$$S^{(1)ab}(p) = -\delta^{ab}\frac{g^2C_V}{16\pi^2}p^2\left[\frac{\pi p}{2m} + \frac{m^2}{p^2} - 1 - \frac{m}{p}\left(\frac{p}{m} - \frac{m}{p}\right)^2\arctan\frac{p}{m}\right]\,, \tag{15}$$

where $p\equiv|p|$.

The one-loop gluon vacuum polarization part can be computed in a similar way. The proper gluon self-energy is determined by gauge symmetry to have the form:

$$\Pi^{ab}_{\mu\nu}(p) = \delta^{ab}\left[\Pi_{\mathrm{O}\mu\nu}(p) + \Pi_{\mathrm{e}\mu\nu}(p)\right]$$
$$= \delta^{ab}\left[\mathrm{sgn}(p)\epsilon_{\mu\nu\rho}p_\rho\Pi_{\mathrm{O}}(p^2) - \frac{1}{m}\left(\delta_{\mu\nu}p^2 - p_\mu p_\nu\right)\Pi_{\mathrm{e}}(p^2)\right]\,, \tag{16}$$

where the subscripts "o" and "e" denote parity-odd and parity-even respectively. The single ghost-loop contribution to the vacuum polarization tensor (Fig. 3b) is

$$-\delta^{ab}\frac{g^2C_V}{16\pi^2 m}\int_{R^3_\epsilon}e^{-ip.x}\partial_\mu\frac{1}{r}\partial_\nu\frac{1}{r}\,. \tag{17}$$

Combining it with the contribution from the singular gluon-loop, we have

$$\Pi_{\mathrm{O}\mu\nu}(p) = i\,\mathrm{sgn}(k)\frac{g^2C_V}{16\pi^2}\int_{R^3_\epsilon}e^{-ip.x}\epsilon_{\mu\nu\rho}\partial^\rho\left[\frac{9}{m^3r^6}(1-e^{-mr})^2\right.$$
$$-\frac{18}{m^2r^5}(1-e^{-mr})e^{-mr} + \frac{1}{mr^4}\left(-1 - \frac{13}{2}e^{-mr} + \frac{33}{2}e^{-2mr}\right) - \frac{1}{2r^3}e^{-mr}$$
$$\left. +\frac{9}{r^3}e^{-2mr} + \frac{m}{4r^2}e^{-mr} - \frac{m^2}{4r}e^{-mr} - \frac{m^2}{4r}e^{-mr} - \frac{m^3}{4}\mathrm{Ei}(-mr)\right]$$
$$= i\,\mathrm{sgn}(k)\frac{g^2C_V}{16\pi^2}\int_{R^3_\epsilon}e^{-ip.x}\epsilon_{\mu\nu\rho}\partial^\rho\left[\frac{3}{8m^3}(\nabla^2)^2\frac{(1-e^{-mr})^2}{r^2}\right.$$
$$+\nabla^2\left(-\frac{1}{2} + \frac{5}{4}e^{-mr} - \frac{3}{4}\frac{e^{-2mr}}{mr^2}\right) + \frac{m}{r^2}\left(-\frac{1}{4}e^{-mr} - 3e^{-2mr}\right)$$
$$\left. -\frac{m^2}{4r}e^{-mr} - \frac{m^3}{4}\mathrm{Ei}(-mr)\right]\,. \tag{18}$$

Again after performing Fourier transformation we obtain

$$\Pi_{\mathrm{o}}(p^2) = \frac{g^2 C_V}{16\pi^2} \left[\left(3\frac{p^3}{m^3} + 5\frac{p}{m} - \frac{m}{p} - \frac{m^3}{p^3} \right) \arctan \frac{p}{m} + \left(-\frac{3}{2}\frac{p^3}{m^3} \right. \right.$$
$$\left. \left. -3\frac{p}{m} + 12\frac{m}{p} \right) \arctan \frac{p}{2m} - \frac{3}{4}\pi\frac{p^3}{m^3} - \pi\frac{p}{m} + \frac{m^2}{p^2} + 2 \right] . \tag{19}$$

The calculation of one gluon-loop contribution to $\Pi_{\mathrm{e}}(p^2)$ is similar but tedious. The result is

$$\Pi_{\mathrm{e}}(p^2) = \int_{R_\epsilon^3} e^{-ip.x} \frac{g^2}{16\pi^2} C_V \left[\frac{3}{2m^4 r^6}(1 - e^{-mr})^2 - \frac{3}{m^3 r^5}e^{-mr}(1 - e^{-mr}) \right.$$
$$+ \frac{1}{m^2 r^4}\left(\frac{1}{8} - \frac{11}{4}e^{-mr} + \frac{33}{8}e^{-2mr}\right) - \frac{1}{mr^3}\left(\frac{7}{4}e^{-mr} + \frac{17}{4}\right)e^{-2mr}$$
$$+ \frac{1}{r^2}\left(\frac{7}{8}e^{-mr} + \frac{1}{4}e^{-2mr}\right) - \frac{1}{r}\left(\frac{m}{2}e^{-2mr} - \frac{7m}{8}e^{-mr}\right)$$
$$\left. - m^2 \mathrm{Ei}(-2mr) - \frac{7m^2}{8}\mathrm{Ei}(-mr) \right]$$
$$= \int_{R_\epsilon^3} e^{-ip.x} \frac{g^2}{16\pi^2} C_V \left\{ \nabla^4 \left[\frac{1}{16}\frac{1}{m^4 r^2}(1 - e^{-mr})^2 \right] \right.$$
$$+ \nabla^2 \left[\frac{1}{m^2 r^2}\left(\frac{1}{16} - \frac{5}{8}e^{-mr} + \frac{9}{16}e^{-2mr}\right) \right] + \frac{1}{r^2}\left(\frac{13}{8}e^{-mr} - 3e^{-2mr}\right)$$
$$\left. - \frac{m}{2r}e^{-2mr} - \frac{7m}{8r}e^{-mr} - m^2 \mathrm{Ei}(-2mr) - \frac{7m^2}{8}\mathrm{Ei}(-mr) \right\}$$
$$= -\frac{g^2 C_V}{32\pi} \left[\left(-8\frac{m^3}{p^3} + 24\frac{m}{p} + \frac{9}{2}\frac{p}{m} - \frac{1}{2}\frac{p^3}{m^3} \right) \arctan \frac{p}{2m} + \right.$$
$$\left(-7\frac{m^3}{p^3} - 13\frac{m}{p} - 5\frac{p}{m} + \frac{p^3}{m^3} \right) \arctan \frac{p}{m} + 11\frac{m^2}{p^2} + \frac{\pi}{4}\frac{p}{m} - \frac{\pi}{4}\frac{p^3}{m^3} + 5 \right] . \tag{20}$$

The next step is to construct the local part of the one-loop effective action and to demonstrate renormalization explicitly. From general principles [25] we know that this construction requires at least one one-loop three-point Green function. Here we choose the one-loop vertex $Ac\bar{c}$, whose Feynman diagrams is shown in Fig. 4.

The amplitudes, which we know is divergentless from dimensional analysis, can be written from Fig. 4 as

$$V_\mu^{abc}(p, q, r) = \frac{1}{2}g^3 C_V f^{abc} \int_x \int_y e^{-i(p.x + q.y)} \left[V_\mu^{(a)}(x, y) + V_\mu^{(b)}(x, y) \right] , \tag{21}$$

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where $p + q + r = 0$. The contribution from Fig. 4a is

$$
\begin{aligned}
V_\mu^{(a)}(x, y) &= iq_\nu \partial_\mu^x S(x - y) \partial_\rho^x S(x) D_{\nu\rho}(y) \\
&= -iq_\nu \frac{1}{(4\pi)^3} \partial_\mu^x \frac{1}{|x - y|} \partial_\rho^x \left[\frac{1}{x} \left(1 - e^{-mx} \right) \right] \\
&\quad \times \left[-i \operatorname{sgn}(k) \epsilon_{\nu\rho\alpha} \partial_\alpha^y - \frac{1}{m} \delta_{\nu\rho} \nabla^{2y} + \frac{1}{m} \partial_\nu^y \partial_\rho^y \right] \frac{1}{y} \left(1 - e^{-my} \right).
\end{aligned}
\tag{22}
$$

For our purpose, that is, to construct the local part of the effective action, only the zero-momentum limit of this amplitude is needed,

$$
V_\mu^{(a)abc}(p, q, r) = -g^3 \frac{C_V}{2} f^{abc} \frac{17}{36} \frac{1}{4\pi} iq_\mu + \cdots.
\tag{23}
$$

The amplitude from Fig. 4b can be reduced to

$$
\begin{aligned}
V_\mu^{(b)abc}&(p, q, r) = \\
\frac{C_V}{2} f^{abc} &\int_x \int_y e^{i(q\cdot y + r\cdot x)} \Big[-i \operatorname{sgn}(k) \epsilon_{\mu\nu\rho} q_\lambda \partial_\sigma^x S(x - y) D_{\nu\lambda}(y) D_{\rho\sigma}(x) \\
&- \frac{1}{m} p_\rho q_\lambda \partial_\sigma^x S(x - y) D_{\mu\lambda}(y) D_{\rho\lambda}(x) + \frac{1}{m} iq_\lambda \partial_\sigma^x S(x - y) \partial_\mu^y D_{\rho\lambda}(y) D_{\rho\sigma}(x) \\
&- \frac{1}{m} iq_\lambda \partial_\sigma^x S(x - y) \partial_\rho^y D_{\mu\lambda}(y) D_{\rho\sigma}(x) + \frac{1}{m} iq_\lambda \partial_\sigma^x S(x - y) D_{\nu\lambda}(y) \partial_\mu^x D_{\nu\sigma}(x) \\
&- \frac{1}{m} iq_\lambda \partial_\sigma^x S(x - y) D_{\nu\lambda}(y) \partial_\nu^x D_{\mu\sigma}(x) + \frac{1}{m} p_\nu q_\lambda \partial_\sigma^x S(x - y) D_{\nu\lambda}(y) D_{\mu\sigma}(x) \Big],
\end{aligned}
\tag{24}
$$

which, after a similar analysis and a lengthy calculation yields the zero-momentum limit

$$
V_\mu^{(b)abc}(p, q, r) = \frac{C_V}{2} f^{abc} \frac{17}{36} \frac{1}{4\pi} iq_\mu + \cdots.
\tag{25}
$$

From Eqs.(21), (23) and (25), we conclude that one-loop $Ac\bar{c}$ vertex takes the form

$$
V_\mu^{abc} = 0 + \cdots.
\tag{26}
$$

This means precisely that $\tilde{Z}(0) = 1$ to one-loop order, $\tilde{Z}(0)$ denotes the $Ac\bar{c}$ vertex renormalization constant defined at $p^2 = 0$. In fact this is the correct result to any order in perturbation expansion for a gauge theory.

IV. ONE-LOOP LOCAL EFFECTIVE ACTION

Having computed the vacuum polarization tensor, the ghost self-energy and the $Ac\bar{c}$ vertex, we are now in a position to derive the local effective action. Our method is the same as that used in [11] for Chern-Simons theory.

We define the generating functional $Z[J, \eta, \bar{\eta}, M, K, L]$ with the external fields K_μ^a and L^a respectively coupled to non-linear BRST transformation products $D_\mu c^a$ and $-g f^{abc} c^b c^c / 2$ as

$$Z[J, \eta, \bar{\eta}, M, K, L] = \int \mathcal{D}X \exp\left[-\left(S + \int_x \left(J_\mu^a A^{\mu a} + \bar{\eta}^a c^a + \bar{c}^a \eta^a\right.\right.\right.$$
$$\left.\left.\left. + B^a M^a + K_\mu^a D^\mu c^a + L^a \left(-\frac{1}{2} g f^{abc} c^b c^c\right)\right)\right)\right], \tag{27}$$

where $X = (A_\mu, B, c, \bar{c})$. The Slavnov-Taylor identity arising from the BRST transformation in (5) is

$$\int_x \left[J_\mu^a \frac{\delta}{\delta K_\mu^a} - \bar{\eta}^a \frac{\delta}{\delta L^a} + \eta^a \frac{\delta}{\delta M^a}\right] Z = 0. \tag{28}$$

In addition, the invariance of $Z[J, \eta, \bar{\eta}, M, K, L]$ under the translations $B^a(x) \to B^a(x) + \lambda^a(x)$, $\bar{c}^a(x) \to \bar{c}^a(x) + \omega^a(x)$ leads respectively to the B-field and antighost field equations:

$$\left[\partial_\mu \frac{\delta}{\delta J_\mu^a} - M^a\right] Z = 0, \tag{29}$$

$$\left[\partial_\mu \frac{\delta}{\delta K_\mu^a} - \eta^a\right] Z = 0. \tag{30}$$

Defining the generating functional for the connected Green function W and that for the one-particle-irreducible Green function Γ (i.e., the quantum effective action) as,

$$W[J, \eta, \bar{\eta}, M, K, L] = -\ln Z[J, \eta, \bar{\eta}, M, K, L]$$

$$\Gamma[A_\mu^a, B^a, c^a, \bar{c}^a, K_\mu^a, L^a] = W[A_\mu^a, B^a, c^a, \bar{c}^a, K_\mu^a, L^a] - \left(A^{\mu a} J_\mu^a + B^a M^a + \bar{\eta}^a c^a + \bar{c}^a \eta^a\right), \tag{31}$$

we obtain the actions of the Slavnov-Taylor identity, the B-field and the anti-ghost field equations on Γ:

$$\partial_\mu A^{\mu a} + \frac{\delta\Gamma}{\delta B^a} = 0 \,, \tag{32}$$

$$\partial_\mu \frac{\delta\Gamma}{\delta K_\mu^a} - \frac{\delta\Gamma}{\delta \bar{c}^a} = 0 \,, \tag{33}$$

$$\int_x \left[\frac{\delta\Gamma}{\delta A^{\mu a}} \frac{\delta\Gamma}{\delta K_\mu^a} - \frac{\delta\Gamma}{\delta c^a} \frac{\delta\Gamma}{\delta L^a} \right] = 0 \,. \tag{34}$$

By a re-definition of Γ:

$$\bar{\Gamma} = \Gamma + \int_x B^a \partial^\mu A_\mu^a \,, \tag{35}$$

these equations become:

$$\frac{\delta\bar{\Gamma}}{\delta B^a} = 0, \qquad \partial_\mu \frac{\delta\bar{\Gamma}}{\delta K_\mu^a} - \frac{\delta\bar{\Gamma}}{\delta \bar{c}^a} = 0 \,, \tag{36}$$

$$\int_x \left[\frac{\delta\bar{\Gamma}}{\delta A^{\mu a}} \frac{\delta\bar{\Gamma}}{\delta K_\mu^a} - \frac{\delta\bar{\Gamma}}{\delta c^a} \frac{\delta\bar{\Gamma}}{\delta L^a} \right] = 0 \,. \tag{37}$$

The first relation in Eq.(36) means that the re-defined action $\bar{\Gamma}$ is independent of B^a and the second relation implies that K_μ^a and $\bar{c}^a$ always appear in $\bar{\Gamma}$ through the combination

$$G_\mu^a(x) = K_\mu^a - \partial_\mu \bar{c}^a \,. \tag{38}$$

Now we introduce the loop-wise expansion for $\bar{\Gamma}$:

$$\bar{\Gamma} = \sum_{n=0}^{\infty} \hbar^n \bar{\Gamma}^{(n)} \,, \tag{39}$$

where $\bar{\Gamma}^{(0)}$ is the classical effective action without the gauge-fixing term $\int_x B^a \partial^\mu A_\mu^a$:

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$$\bar{\Gamma}^{(0)} = -i \operatorname{sgn}(k) \int_x \epsilon^{\mu\nu\rho} \left(\frac{1}{2} A_\mu^a \partial_\nu A_\rho^a + \frac{1}{3!} g f^{abc} A_\mu^a A_\nu^b A_\rho^c \right)$$
$$+ \frac{1}{4m} \int_x F_{\mu\nu}^a F^{\mu\nu a} + \int_x \left[G_\mu^a D^\mu c^a + L^a \left(-\frac{1}{2} g f^{abc} c^b c^c \right) \right]. \tag{40}$$

Substituting this expansion into Eq.(37) and comparing the coefficients of the $\hbar^0$ and $\hbar^1$ terms lead to

$$\int_x \left[\frac{\delta \Gamma^{(0)}}{\delta A^{\mu a}} \frac{\delta \Gamma^{(0)}}{\delta G_\mu^a} - \frac{\delta \Gamma^{(0)}}{\delta c^a} \frac{\delta \Gamma^{(0)}}{\delta L^a} \right] = 0 \tag{41}$$

and

$$\Delta \Gamma^{(1)} = 0, \tag{42}$$

where we have used the relation

$$\frac{\delta}{\delta G_\mu^a} = \frac{\delta}{\delta K_\mu^a}. \tag{43}$$

Δ, the linear Slavnov-Taylor operator

$$\Delta = \int_x \left[\frac{\delta \bar{\Gamma}^{(0)}}{\delta A^{\mu a}} \frac{\delta}{\delta G_\mu^a} + \frac{\delta}{\delta A^{\mu a}} \frac{\delta \bar{\Gamma}^{(0)}}{\delta G_\mu^a} - \frac{\delta \bar{\Gamma}^{(0)}}{\delta c^a} \frac{\delta}{\delta L^a} - \frac{\delta}{\delta c^a} \frac{\delta \bar{\Gamma}^{(0)}}{\delta L^a} \right], \tag{44}$$

is the quantum analogue of the classical BRST operator and is nilpotent :

$$\Delta^2 = 0. \tag{45}$$

Now we follow the method of [11, 20] to find the solution to Eq.(42). From the requirement of zero ghost-number and mass dimension 3 we determine the general form of one-loop effective action to be that* [1]

---

[1]Rigorously speaking, the one-loop local effective action given here is not perfect, it should contain other $(1/m)^n$ $(n \geq 1)$ dependent higher covariant derivative terms such as $\frac{1}{m^{2n+1}} \int_x F_{\mu\nu} (D^2)^n F^{\mu\nu}$, $\frac{1}{m} \Delta \int_x \epsilon^{\mu\nu\rho} G_\mu^a$ and $\frac{1}{m} \Delta \int_x (L^a c^a)(\bar{c}^b c^b)$ etc. They also have correct mass dimension and ghost number. However since in this section our aim is at the large m-limit, we have put these $1/m$ terms out of consideration

$$\Gamma^{(1)} = \alpha_1 \left[-i \operatorname{sgn}(k) \int_x \epsilon^{\mu\nu\rho} \left(\frac{1}{2} A_\mu^a \partial_\nu A_\rho^a + \frac{1}{3!} g f^{abc} A_\mu^a A_\nu^b A_\rho^c \right) \right]$$

$$+ \alpha_2 \frac{1}{4m} \int_x F_{\mu\nu}^a F^{\mu\nu a} + \Delta \int_x [\beta_1 G^{\mu a} A_\mu^a + \beta_2 L^a c^a]$$

$$= -i \operatorname{sgn}(k)(\alpha_1 + 2\beta_1) \int_x \frac{1}{2} \epsilon^{\mu\nu\rho} A_\mu^a \partial_\nu A_\rho^a$$

$$- i \operatorname{sgn}(k)(\alpha_1 + 3\beta_1) \int_x \frac{1}{3!} \epsilon^{\mu\nu\rho} f^{abc} A_\mu^a A_\nu^b A_\rho^c$$

$$+ (\alpha_2 + 2\beta_1) \frac{1}{4m} \int_x (\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)$$

$$+ (\alpha_2 + 3\beta_1) \frac{1}{2m} \int_x g f^{abc} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A^{\mu b} A^{\nu c}$$

$$+ (\alpha_2 + 4\beta_1) \frac{1}{4m} \int_x g^2 f^{eab} f^{ecd} A_\mu^a A_\nu^b A^{\mu c} A^{\nu d} + \beta_1 \int_x G_\mu^a \partial^\mu c^a$$

$$+ \beta_2 \int_x G_\mu^a D^\mu c^a - \beta_2 \int_x \frac{1}{2} g f^{abc} c^b c^c , \tag{46}$$

where α_i and β_i are constant coefficients. In comparison with CS [11, 20], we find that the formal large-m limit of above effective action has the same form as that in refs. [11, 20], i.e., the difference lies only in the mass dependent terms. By using the results given in the last section and choosing the renormalization point at $|p| = 0$, we can determine the values of the parameters as follows:

$$\alpha_1 = \frac{g^2 C_V}{4\pi}, \quad \alpha_2 = -\frac{g^2 C_V}{32\pi}, \quad \beta_1 = -\frac{g^2 C_V}{16\pi}, \quad \beta_2 = 0 . \tag{47}$$

Thus, up to one-loop the explicit local effective action is:

$$\Gamma_{\text{local}} = \Gamma^{(0)} + \Gamma_{\text{local}}^{(1)}$$

$$= \left(1 + \frac{1}{4\pi} g^2 C_V \right) [-i \operatorname{sgn}(k)] \int_x \epsilon^{\mu\nu\rho} \left(\frac{1}{2} A_\mu^a \partial_\nu A_\rho^a + \frac{1}{3!} A_\mu^a A_\nu^b A_\rho^c \right)$$

$$+ \frac{1}{4m} \left(1 - \frac{1}{32\pi} g^2 C_V \right) \int_x F^{\mu\nu a} F_{\mu\nu}^a$$

$$- \Delta \left(\frac{1}{16\pi} g^2 C_V \int_x G_\mu^a A_\mu^a + \int_x L^a c^a \right) + \int_x B^a \partial^\mu A_\mu^a$$

$$= \left(1 + \frac{1}{4\pi} g^2 C_V \right) [-i \operatorname{sgn}(k)] \int_x \epsilon^{\mu\nu\rho} \left(\frac{1}{2} A_\mu^a \partial_\nu A_\rho^a + \frac{1}{3!} A_\mu^a A_\nu^b A_\rho^c \right)$$

$$+ \frac{1}{4m} \left(1 + \frac{9}{32\pi} g^2 C_V \right)^{-1} \int_x F^{\mu\nu a} F_{\mu\nu}^a$$

$$- \Delta \left(\frac{1}{16\pi} g^2 C_V \int_x G^{\mu a} A_\mu^a + \int_x L^a c^a \right) + \int_x B^a \partial^\mu A_\mu^a . \tag{48}$$

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Finally the one-loop effective action of CS can be easily obtained by taking the large-mass limit $m\to\infty$. Obviously the wave-function renormalization constants are

$$Z_A = Z_B^{-1} = Z_G^{-1} = 1 - \frac{1}{16\pi}g^2 C_V, \qquad Z_L = Z_C^{-1} = 1\,. \tag{49}$$

This result can be cast into $k-$shift form, i.e.,

$$k \to k + \mathrm{sgn}(k)C_V\,. \tag{50}$$

V. GAUGE FIELD SELF-ENERGY IN THREE-DIMENSIONAL QED

In the Appendix are given some formulas which are useful for computation in the study of any three-dimensional theory in coordinate space. Here, by the way, we would like to point out that, as an example, using these formulas we can get the one-loop self-energy part for the gauge field in three-dimensional massive QED an analytic expression whose integral form was given in ref. [5],

$$\Pi_{ij}(p) = -\frac{e^2}{8\pi}\left(\delta_{ij}p^2 - p_ip_j\right)\left[\frac{2m}{p^2} + \left(\frac{1}{p} - \frac{4m^2}{p^3}\right)\arctan\frac{p}{2m}\right]$$
$$-me^2\epsilon_{ijk}\frac{1}{2\pi p}\arctan\frac{p}{2m}\,, \tag{51}$$

where the notation is the same as that in ref. [5].

VI. SUMMARY

We carried out the one-loop calculation of the topologically massive Yang-Mills theory and Chern-Simons theory in coordinate space using the method of differential renormalzation. Our calculation shows that the method is very powerful and is especially suited for quantum field theories in three dimensions. The results we obtained on TMYM coincide with those of ref. [9], which used the method of dimensional

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regularization. However, as was pointed out in ref. [9], the calculus of dimensional regularization for a theory in three dimensions is subtle and perhaps even problematic, not least because of the need for a dimensional continuation of the antisymmetric tensor $\epsilon_{\mu\nu\rho}$; it is not known to what extent the calculated renormalization of a field theory such as TMYM, whose property is closely tied to the dimension of space-time, could be an artifact of this continuation. In differential renormalzation there is not such an ambiguity because one does not change the dimension of space-time so there is no need for a continuation of the antisymmetric tensor. It is therefore reassuring that the two sets of results agree. For Chern-Simons field theory our result shows the shift k to $k + \mathrm{sgn}(k)C_V$, which coincides with the case of scalar regulator of ref. [19].

REFERENCES

[1] D.Z. Freedman, K. Johnson and J.I. Latorre, Nucl. Phys. **B371** (1992) 353.

D. Z. Freedman, *"Differential Regularization and Renormalization: Recent Progress"* in *Proceedings of the Stone Brook Conference on String and Symmetries.*

[2] P.E. Haagensen and J.I. Latorre, Phys. Lett. **B283** (1992) 293.

[3] P.E. Haagensen, Mod. Phys. Lett. **A7** (1992) 893.

[4] P.E. Haagensen and J.I. Latorre, Ann. Phys. **221** (1993) 77.

[5] R. Munoz-Tapia, Phys. Lett. **B295** (1992) 95.

[6] G. Dunne and N. Rius, Phys. Lett. **B293** (1992) 367.

[7] D.Z. Freedman, K. Johnson, R. Munoz-Tapia and X. Vilasis-Cardona, Nucl. Phys. **B395** (1993) 454.

[8] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48** (1982) 975; Phys. Rev. **D23** (1981) 2291; Ann. Phys. **140** (1982) 372; Ann. Phys. **185** (1988) 406.

[9] R.D. Pisarski and S. Rao, Phys. Rev. **D32** (1985) 2081; E.R. Bezerra De Mello, Ann. Phys. **185** (1988) 401.

[10] L. Alvarez-Gaume, J.M.F. Labastida and A.V. Ramallo, Nucl.Phys. **B334** (1990) 103.

[11] G. Giavarini, C.P. Martin and F. Ruiz. Ruiz. Nucl. Phys. **B381** (1992) 222; C.P. Martin, Phys. Lett. **B241** (1990) 513.

[12] W. Chen, G.W. Semenoff and Y.S. Wu. Mod. Phy. Lett. **A5** (1990) 1833; Phys. Rev. **D46** (1992) 5521.

[13] W.F. Chen and Z.Y. Zhu, J. Phys. A: Math. Gen. **27** (1994) 1781.

[14] D. Birmingham, M. Rakowsky and G. Thompson, Phys. Lett. **B251** (1990) 121; E. Witten, Comm. Math. Phys. **121** (1989) 351.

[15] M. Asorey and F. Falceto, Phys. Lett. **B241** (1990) 31.

[16] E. Guadagnini, M. Martellini and M. Mintchev, Phys. Lett. **B227** (1989) 111.

[17] H.C. Kao, K. Lee and T. Lee, Phys. Lett. **B373** (1996) 94.

[18] S. Axelrod and I.M. Singer, *Chern-Simons Perturbation Theory*, In ∗ New York 1991, Proceedings of Differential Geormetric Method in Theoretical Physics, **1** 3; **hep-th/9110056**.
S. Axelrod and I.M. Singer, *Chern-Simons Perturbation Theory* **II**, **hep-th/9304987**.

[19] M. Asorey, I. Falceto, J.L. López and G. Luzón, Phys. Rev. **D49** (1993) 5377; Nucl. Phys. **B429** (1994), 344.

[20] G. Giavarini, C.P. Martin and F. Ruiz Ruiz, NIKHEF-H 93-05, *Physical meaning and not so meaningful symmetries in Chern-Simons field theory* (To appear in Phys. Rev. D).
G. Giavarini, LPTHE 92-42, "*Low dimensional topology and quantum field theory*", (talk given at the NATO AWR, 6-13 September, 1992, Cambridge (UK)); NIKHEF-H 93-05, "*The universality of the shift of the Chern-Simons parameter for a General Class of BRST invariant regularization*".

[21] N.N. Bogoliubov and D.V. Shirkov, "*Quantum Fields*" (Benjamin, Reading, 1983).

[22] V.A. Smirnov, Nucl. Phys. **B427** (1994) 325; Z. Phys.**C67**, (1995) 531.

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[23] D. Birmingham, M. Rakowsky and G. Thompson, Nucl. Phys. **B234** (1990) 367; F. Delduc, F. Gieres and S.P. Sorella, Phys. Lett. **B225** (1989) 367; P.H. Damgarrd and V.O. Rivelles, Phys. Lett. **B245** (1989) 48.

[24] D. Daniel and N. Dorey, Phys. Lett. **B246** (1990) 82; N. Dorey, Phys. Lett. **B 246** (1990) 87; F. Delduc, C. Lucchesi, O. Piguet and S.P. Sorella, Nucl. Phys. **B346** (1990) 313.

[25] B.W. Lee and J. Zinn-Justin, Phys. Rev. **D5** (1972) 3121.

J.C. Collins, *"Renormalization"* (Cambridge University Press, Cambridge, 1987).

APPENDIX A:

1. Differential formulas

Defining

$$f^{(n)} \equiv \left(\frac{1}{r}\frac{d}{dr}\right)^n f(r)\,, \qquad f(r) = \frac{1 - e^{-mr}}{r}\,, \tag{A1}$$

and denoting $\partial_1 = \partial_{i_1}$, $x_1 = x_{i_1}$ etc, we have

$$\partial_1\partial_2\cdots\partial_{2n}f = (\delta_{12}\delta_{34}\cdots\delta_{2n-1,2n} + \text{permutations})f^{(n)}$$

$$+(x_1x_2\delta_{34}\delta_{56} + \text{permutations})f^{(n+1)}$$

$$+(x_1x_2x_3x_4\delta_{56}\delta_{78}\cdots\delta_{2n-1,2n} + \text{permutations})f^{(n+2)}$$

$$+\cdots+(x_1x_2\cdots x_{2n})f^{(2n)}\,, \tag{A2}$$

$$\partial_1\partial_2...\partial_{2n+1}f = (x_1\delta_{23}\delta_{45}\cdots\delta_{2n,2n+1} + \text{permutations})f^{(n+1)}$$

$$+(x_1x_2x_3\delta_{45}\cdots\delta_{2n,2n+1} + \text{permutations})f^{(n+2)}$$

$$+\cdots+(x_1x_2\cdots x_{2n+1})f^{(2n+1)}\,. \tag{A3}$$

Some examples are:

$$\partial_i f = x_i f^{(1)}\,,$$

$$\partial_i\partial_j f = \delta_{ij}f^{(1)} + x_i x_j f^{(2)}\,,$$

$$\partial_i\partial_j\partial_k f = (x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij})f^{(2)} + x_i x_j x_k f^{(3)}\,. \tag{A4}$$

2. Fourier transforms of some functions

$$\int_x \frac{1}{x^2} e^{ip.x} = \frac{2\pi^2}{|p|} \,.$$

$$\int_x \frac{e^{-nmx}}{x} e^{ip.x} = \frac{4\pi}{n^2 m^2 + p^2} \,.$$

$$\int_x \frac{e^{-nmx}}{x^2} e^{ip.x} = \frac{4\pi}{p} \arctan \frac{p}{nm} \,.$$

$$\int_x \mathrm{Ei}(-nmx) e^{ip.x} = -\frac{8\pi}{p^3} \left[\frac{\pi}{4} - \frac{1}{2} \arctan \frac{mn}{p} - \frac{1}{4} \frac{mnp}{p^2 + m^2 n^2} \right] \,. \tag{A5}$$

3. Integrals over R_ϵ^3

Recall that R_ϵ^3 is R^3 excluding a small ball B_ϵ of radius ϵ about the origin.

$$\int_{R_\epsilon^3} e^{-ip.x} \partial_\mu f(x) = -if(\epsilon) 4\pi \frac{p_\mu}{p} \frac{d}{dp} \left[\frac{\sin(p\epsilon)}{p} \right] + ip_\mu \int_x e^{-ip.x} f(x) \,. \tag{A6}$$

$$\int_{R_\epsilon^3} e^{-ip.x} \partial_\mu \partial_\nu f(x) = 4\pi \frac{1}{\epsilon} \frac{d}{dx} f(x)|_{x=\epsilon} \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p_\nu} \left[\frac{\sin(p\epsilon)}{p} \right]$$
$$+ 4\pi f(\epsilon) \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p_\nu} \left[\frac{\sin(p\epsilon)}{p} \right] - p_\mu p_\nu \int_x e^{-ip.x} f(x) \,. \tag{A7}$$

$$\int_{R_\epsilon^3} e^{-ip.x} \partial^2 \partial_\mu f(x) = -i(4\pi) \partial^2 f(x)|_{x=\epsilon} \frac{\partial}{\partial p_\mu} \left[\frac{\sin(p\epsilon)}{p} \right]$$
$$+ i4\pi p_\mu p_\alpha \epsilon f(\epsilon) \frac{\partial}{\partial p_\alpha} \left[\frac{\sin(p\epsilon)}{p} \right]$$
$$- i4\pi \epsilon p_\mu \frac{d}{dx} f(x)|_{x=\epsilon} \frac{\sin(p\epsilon)}{p} - ip_\mu p^2 \int_x e^{-ip.x} f(x). \tag{A8}$$

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$$\int_{R_\epsilon^3} e^{-ip.x} \partial_\mu \partial_\nu \partial_\rho f(x) = i4\pi \delta_{\nu\rho} \frac{\partial}{\partial p_\mu} \frac{\sin(p\epsilon)}{p\epsilon} \left[\frac{1}{x} \frac{d}{dx} f(x) \right]_{x=\epsilon}$$

$$+ i4\pi \frac{\partial^3}{\partial p_\mu \partial p_\nu \partial p_\rho} \frac{\sin(p\epsilon)}{p\epsilon} \frac{1}{x} \frac{d}{dx} \left[\frac{1}{x} \frac{d}{dx} f(x) \right]_{x=\epsilon}$$

$$- ip_\mu \int_x e^{-ip.x} \partial_\nu \partial_\rho f(x). \tag{A9}$$

4. Short-distance expansions

$$\frac{\partial}{\partial p_\mu} [\frac{\sin(p\epsilon)}{p}] = p_\mu \left[-\frac{\epsilon^3}{3} + \frac{p^2\epsilon^5}{30} - \frac{p^4\epsilon^7}{30} + \mathcal{O}(\epsilon) \right]. \tag{A10}$$

$$\frac{\partial^2}{\partial p_\mu \partial p_\nu} \left[\frac{sin(p\epsilon)}{p} \right] = \delta_{\mu\nu} \left[-\frac{\epsilon^3}{3} + \frac{p^2\epsilon^5}{30} - \frac{p^4\epsilon^7}{30} + \mathcal{O}(\epsilon) \right]$$

$$- p_\mu p_\nu \left[-\frac{\epsilon^5}{15} - \frac{p^2\epsilon^7}{210} + \frac{p^4\epsilon^9}{7560} + \mathcal{O}(\epsilon) \right]. \tag{A11}$$

FIGURES

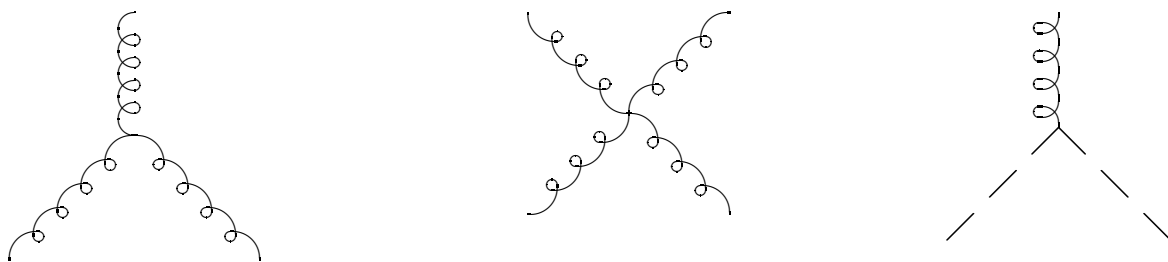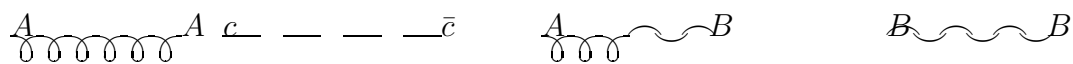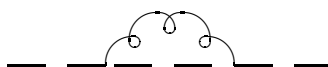A ⌇⌇⌇⌇⌇ A c — — — $\bar{c}$ A ⌇⌇⌇ B B ⌇⌇⌇ B

FIG. 1. Feynman Rules

FIG. 2. Ghost self-energy
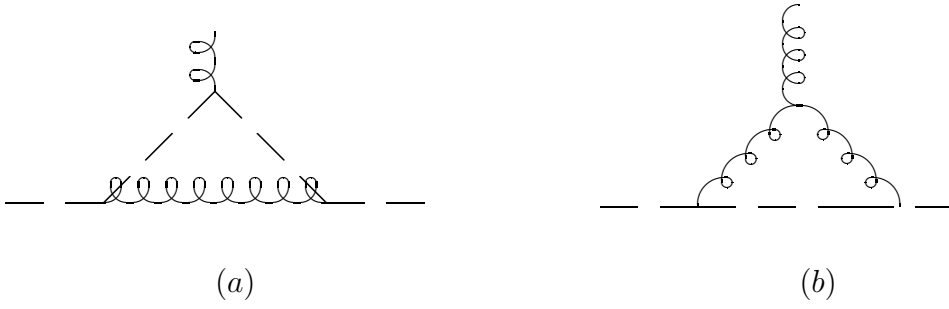
(a) (b)
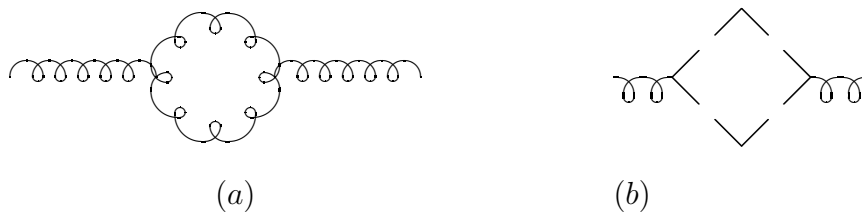
FIG. 3. Vacuum polarization tensor

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(a) (b)

FIG. 4. One-loop ghost-gluon vertex